MTH102
Engineering Mathematics II
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Chapter 1.4 Combinatorics

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Chapter 1.4 Permutations and Combinations

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1.4.1 Permutations

A *permutation* of objects is an arrangement of these objects in a row in some order. Order counts.

We distinguish between choosing

- *with repetition*: the same element can be chosen again
- *without repetition*: only different elements can be chosen

1.4.1 Permutations

For three distinct letters a, b, c how many ways are there to form a three-letter word?

Solution

i. with repetition (each time three choices) 27 ways

$$aaa, aab, aac, aba, abb, abc, aca, acb, acc$$
$$baa, bab, bac, bba, bbb, bbc, bca, bcb, bcc$$
$$caa, cab, cac, cba, cbb, cbc, cca, ccb, ccc$$

ii. without repetition (3 choices, then 2, last 1) 6 ways

$$abc, acb, bac, bca, cab, cba$$

1.4.1 Permutations with repetition

We can choose *k* objects from *n*. For example *k* people can choose one of three (*n* = 3) ice-cream flavours (people are allowed to choose the same flavour).

Assume there is only one person (*k* = 1). There are 3 possible choices.

For two people (*k* =2) the second person also has three choices. So the total possible choices are $3 * 3 = 3^2$. For *k* = 3 people the choices are $3 * 3 * 3 = 3^3$, and so on.

Repeating the reasoning, for generic *k* objects from n, the total number of permutations with repetition is n^k.

1.4.1 Permutations with repetition: example 1

For the permutation of three distinct letters a, b, c in three-letter words with repetition, we have $n = 3$ and $k = 3$. So we have $3^3 = 27$ ways

Solution

$$aaa, aab, aac, aba, abb, abc, aca, acb, acc$$
$$baa, bab, bac, bba, bbb, bbc, bca, bcb, bcc$$
$$caa, cab, cac, cba, cbb, cbc, cca, ccb, ccc$$

1.4.1 Permutations (without repetition)

The number of *permutations without repetition (n objects in n),* or ordered arrangements, of n distinct objects is

$n(n-1)(n-2)\cdots 2\cdot 1 = n!$ (*read n-factorial) [0! = 1]*

Proof

1st	2nd	3rd	$\cdots$	$\cdots$	$(n-1)th$	n-th
n choices	(n-1) choices	(n-2) choices	$\cdots$	$\cdots$	2 choices	1 choice only I left

Total $n(n-1)(n-2)\cdots 2 * 1 = n!$

■

1.4.1 Permutations (without repetition): example 2

For the permutation of three distinct letters a, b, c in three-letter words without repetition, we have $n = 3$ and $k = 3$. So we have $3! = 2 * 3 = 6$ ways

$$abc, acb, bac, bca, cab, cba$$

1.4.1 Permutations without repetition: example 3

We can choose k objects from n. For example an ice-cream seller can display 5 (*n* = 5) flavours in different order.

The first flavour (*k* = 1) can be anyone. So there are 5 (=*n*) possible choices.

The second flavour (*k* = 2) can be anyone of the remaining 4. So, there are $5 * 4 = 20$ (=n* (n-1)) possible choices. The third flavour (*k* =3) can be chosen from the remaining 3 flavours. So, for there are $5 * 4 * 3 = 60$ (=n* (n-1)*(n-2)) possible choices. And so on.

For n objects from n the total number of combinations without repetition is $n! = 120$.

1.4.1 Permutations without repetition of k objects from n

The number of *permutations without repetition*, or ordered arrangements, of k from n distinct objects is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Proof

1st	2nd	3rd	⋯	⋯	⋯	kth
n choices	(n-1) choices	(n-2) choices	⋯	⋯	⋯	(n-k+1) choices

Total $n(n-1)(n-2)\cdots(n-k+1)$ ■

Note: Can write $P(n,k) =_n P_k$!

1.4.1 Permutations (without repetition): example 4

For the permutation of three distinct letters a, b, c in two-letter words without repetition, we have $n = 3$ and $k = 2$. So we have $\frac{3!}{3-2} = \frac{2*3}{1} = 6$ ways

$$ab, ba, ac, ca, bc, cb$$

1.4.1 Permutations: example 5

Example 6

How many nonrepeating three-digit numbers can be written using digits from the set $\{3, 4, 5, 6, 7, 8\}$?

Solution

Repetitions are not allowed since numbers are nonrepeating, e.g. 448 is not allowed. Also, order is important, e.g. 476 and 467 are distinct.

No. of ways $= P(6, 3) = 6 \cdot 5 \cdot 4 = 120$ ■

1.4.1 Permutations (binning)

If n objects consisting of c classes of identical objects with size $n_1, n_2, \cdots, n_c$ (such that $n_1 + n_2 + \cdots + n_c = n$), then the number of permutations of these n objects is $P\big((n_1, n_2, \dots, n_c), n\big) = \frac{n!}{n_1! n_2! \cdots n_c!}$. Also called multinomial coefficient $\binom{n}{n_1 n_2 \cdots n_c}$

Proof

The n_1 identical objects in class 1 make $n_1!$ permutations collapse into a single permutation (those in which class 1 objects occupy the same n_1 positions) etc, so that this follows from the previous proof.

■

<u>In this case the order of identical objects cannot be recognized!</u>

1.4.1 Permutations (binning): example 6

Example 7

If a box contains 3 red and 2 blue balls, in how many ways can we arrange all the balls?

Solution

There $n = 5$ balls, $n_1 = 3$ red and $n_2 = 2$ blue.

The number of permutations $= \frac{5!}{3!2!} = 10$. ■

1.4.1 using your calculator for combinatorics

Using the $_nP_r$ and $_nC_r$ function is safer because it avoids memory overflow.

1.4.1 Permutations: problem 1

Suppose certain account numbers are to consist of two letters followed by four digits and then three more letters, where repetitions of letters or digits are not allowed *within* any of the three groups, but the last group of letters may contain one or both of those used in the first group. How many such account numbers are possible? (there are 26 letters in the alphabet)
Example EX 2578 ABE

1.4.2 Combinations

Suppose we are now interested in the number of subsets of size r, where $r \leq n$, that can be chosen from n distinct objects. Order does not count.

The order of elements in each subset makes no difference. Subsets in this context are called *combinations* which we denote as $C(n,r)$ or ${}_nC_r$. This is also called binomial coefficient

$$C(n,r) = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

1.4.2 Combinations

Proof

Let us consider the permutation of r objects that can be selected from n distinct objects. Since the number of subsets of r elements is $C(n, r)$ and the elements in each subset can be arranged in $r!$ ways, then $P(n, r) = r!\, C(n, r)$.

Rearranging,

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!\,(n-r)!} \quad \blacksquare$$

1.4.2 Combinations

Note 1

Combinations are applied when

1. order is not important, and
2. repetitions are not allowed, and
3. we choose from set of distinct items

Note 2

$\binom{0}{0}$ is defined as 1, $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{k} = \binom{n}{n-k}$

1.4.2 Combinations: example 7

Example 7

How many ways can two slices of pizza be chosen from a plate containing one slice each of pepperoni, sausage, mushroom and cheese pizza?

Solution

In choosing the slices of pizza, order is not important. This arrangement is a combination.

No. of ways$= C(4,2) = \frac{4!}{2!2!} = 6.$ ■

1.4.2 Combinations: example 8

Example 8

In a group, there are 3 men and 2 women. In how many ways are there to form a 3-member committee so that exactly one woman is on the committee?

Solution

Order does not matter. The task has two parts:

i. Choose one woman, and
ii. Choose two men.

1.4.2 Combinations: example 8 ct.d

Solution ct.d

One women can be chosen from $C(2,1) = 2$ ways. Two men can be chosen from $C(3,2) = 3$

ways.

No. of ways$= 2 \cdot 3 = 6$ ■

1.4.2 Combinations: problem 2

There are 26 letters in the alphabet. Suppose certain account numbers are to consist of two letters followed by four digits and then three more letters, where order doesn't count but repetitions of letters or digits are not allowed *within* any of the three groups, but the last group of letters may contain one or both of those used in the first group. How many such account numbers are possible?

Example EX 2578 ABE ≡ XE 7258 BEA

1.4.2 Combinations: problem 3

There 7 men and 5 women available to form a committee with 2 women and 3 men.

1. How many different committees are possible?

2. How many different committees are possible if two of the men refuse to serve together?

1.4.3 Probability of Counting: example 9 (1)

Permutations and combinations can be used in finding probabilities.

Example 9

Compute the probability of obtaining only three ‘6’s in rolling a fair dice 4 times.

1.4.3 Probability of Counting: example 9 (2)

Solution (3 '6's in 4 rolls) (666x), (66x6), (6x66), (x666)

There are 4 trials and 6 possible results in each. The number of ways of arranging the three '6' is $\frac{4!}{3!1!} = 4$, for each one of these there 5 other possible results, so $5\frac{4!}{3!1!} = 20$. The total number of possible results is $6^4 = 1296$.

So the required probability is $\frac{4\times5}{1296} = \frac{5}{324} = 0.0154$ ■

1.4.3 Probability of Counting: example 9 (3)

Solution (3 '6's in 4 rolls) (666x), (66x6), (6x66), (x666)

Another way of thinking about this problem

There are 4 trials and 6 possible results in each. In one trial the result must not be '6', so there are 5 possible results $\{1, 2, 3, 4, 5\}$.
The non-6 can happen in any of the 4 trials, so there are 5*4 = 20 possible permutations (remember that the three '6's are non distinguishable).

The total number of possible results is $6^4 = 1296$.

So the required probability is $\frac{4 \times 5}{1296} = \frac{5}{324} = 0.0154$ ■

1.4.3 Probability of Counting : example 9 (4)

Another way of looking at this problem is to consider the probabilities of getting a 6 (event A), $P(A) = 1/6$. The number of ways of getting three '6' is $\frac{4!}{3!1!} = 4$ ($AAA\bar{A}, AA\bar{A}A, A\bar{A}AA$ and $\bar{A}AAA$).

Each ordered arrangement has a probability of

$$\mathrm{P(A)P(A)P(A)P(}\bar{A}) = \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right) = \left(\frac{5}{1296}\right)$$ of occurring.

Required probability $= 4 \times \frac{5}{1296} = \frac{5}{324}$ ■

This is the way we will model probabilities later on.

1.4.3 Probability of Counting: example 10

Example 10

Given a class of 12 girls and 10 boys.

i. In how many ways can be a committee of five consisting of 3 girls and 2 boys be chosen?
ii. What is the probability that a committee of five, chosen at random from the class, consists of 3 girls and 2 boys?
iii. How many of the possible committees of five consists only of girls?
iv. What is the probability that a committee of five, chosen at random from the class, consists of at least one boy?

1.4.3 Probability of Counting : example 10 solutions

Solution

i. No. of ways to choose 3 girls= $\binom{12}{3} = 220$

No. of ways to choose 2 boys= $\binom{10}{2} = 45$

Total number of ways= $220 \cdot 45 = 9900$

ii. Without constraint, the total number of ways= $\binom{22}{5} = 26334$

Required probability= $\frac{9900}{26334} = \frac{50}{133}$

1.4.3 Probability of Counting : example 10 solutions

Solution ct.d

iii. No. of ways to choose 5 girls= $\binom{12}{5} = 792$

iv. No. of ways to form committees with at least one boy. These all the possibilities minus those that contain only girls

$$= 26334 - 792 = 25542$$

Required probability= $\frac{25542}{26334} = \frac{129}{133}$ ■

1.4.3 Probability of Counting: problem 3

Find the probability of drawing 2 Kings from a deck of 52 cards? (There are 52 cards of four suits, 4 Kings, 4 Queens, *etc*.)

[Hint: think in terms of number of combinations of two cards]

1.4.4 Permutations again

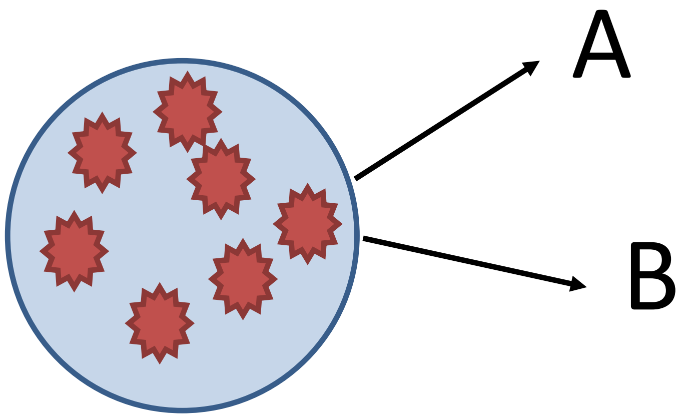
Example 3

How many ways are there to

i. distribute n distinct sweets

ii. distribute n identical sweets

between person A and B?



Note: here the number of sweets given to each person is not fixed. The distribution could be (0, n), (1, n - 1), …, (n, 0)

1.4.4 Permutations again

i. (different sweets)

For each sweet, select either A or B to be its owner. Assign to each sweet the value 0 if it goes to A and 1 if it goes to B. So, the problem is: choose n times 0 or 1. This is a simple permutation with repetition.

For n sweets, there are $\binom{2}{1}^n = 2^n$ ways

This is called size of the power set

1.4.4 Permutations again

ii. (identical sweets)
When the sweets are identical, we cannot consider combinations. Instead we consider: how many sweets can we give to A? $\{0, 1, 2, \dots, (n-1), n\}$. In total $(n+1)$ possibilities.

Think of this as an $(n+1)$-long binary string consisting of n '0' and a '1'. We consider the zeros to the left of '1' to be sweets for A and zeros to the right of '1' to be sweets for B.

Number of ways of arranging a 1 in (n+1) is $= \frac{(n+1)!}{n!1!} = n + 1$ ■

(100000) 0 to A, (010000) 1 to A, ……, (000001) 5 to A

1.4.4 Summary

- 1.4.1 Permutations order, $\left[n!, \frac{n!}{(n-k)!}, \frac{n!}{n_1!n_2!\ldots n_{k!}}\right]$
- 1.4.2 Combinations no order, $\left[\binom{n}{k} = \frac{n!}{(n-k)!k!}\right]$
- 1.4.3 Probability of Counting, $\left[\frac{restricted\ number}{total\ number}\right]$
- 1.4.4 Permutations no fixed size $[2^n$, *(n + 1)*]